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by

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I. Introduction

In this paper we present results related to the smoothing problem and related generalized Riccati equations for the two-point boundary value descriptor system (TPBVDS)

$$Ex(k+1) = Ax(k) + Bu(k) \quad (1)$$

$$V_i x(0) + V_f x(N) = v \quad (2)$$

$$y(k) = Cx(k) \quad (3)$$

where E, A, V_i and V_f are possibly singular nxn matrices, and B and C are nxm and pxn matrices respectively.

II. System Theory for TPBVDSs

In [1-2] we develop a basic theory for (1)-(3). Many of the aspects of this theory have a similar flavor to that in [4-5], although the possible singularity of E and A creates some significant differences. As discussed in [1,2], when (1)-(2) is well-posed, we can assume that it is in standard form, i.e. for some constants α and β

$$\alpha E+\beta A = I \quad (4)$$

and

$$V_i E^N + V_f A^N = I \ . \quad (5)$$

As in [4-5], x(k) can be decomposed into an *outward process* z_o and an *inward process* z_i. The outward process z_o is defined as

$$z_o(k,\ell) = E^{\ell-k}x(\ell) - A^{\ell-k}x(k) \qquad k<\ell. \quad (6)$$

By eliminating x's in (6), we find that $z_o(k,\ell)$ is only a function of the inputs inside the interval $[k,\ell]$. Also note that z_o does not depend in any way on the boundary matrices V_i and V_f. The expression for the inward process z_i is in general very complex, although in the so-called stationary case there is a simple expression for z_i [1].

The system (1)-(2) is *strongly reachable* on $[k,\ell]$ if the map from $\{u(m): m\in[k,\ell-1]\}$ to $z_o(k,\ell)$ is onto. System (2.1) is called *strongly reachable* if it is reachable on some $[k,\ell]$.

Theorem 1:
The following statements are equivalent
a) System (1)-(2) is strongly reachable.
b) The strong reachability matrix

$$R = \left[A^{n-1}B \,\vdots\, EA^{n-2}B \,\vdots\, \cdots \,\vdots\, E^{n-1}B\right] \quad (7)$$

has full rank.
c) The matrix $[sE-tA \vdots B]$ has full rank for all $(s,t)\neq(0,0)$.
d) The state x(i) where $i\in[n,N-n]$ can be made arbitrary by proper choice of the inputs $u(j)$: $j\in[i-n,i+n-1]$ with all other inputs and the boundary value v set to zero, and for all pair of matrices V_i and V_f in standard form.

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The system (1)-(3) is *strongly observable* on $[k,\ell]$ if the map $z_i(k,\ell)\rightarrow\{y(m): m\in[k,\ell]\}$ is one to one. System (1)-(3) is called *strongly observable* if it is observable on some $[k,\ell]$.

Theorem 2:
The following statements are equivalent
a) System (1)-(3) is strongly observable.
b) The strong observability matrix

$$\begin{bmatrix} CA^{n-1} \\ CEA^{n-2} \\ \vdots \\ CE^{n-1} \end{bmatrix} \quad (8)$$

has full rank.
c) The matrix $\begin{bmatrix} sE-tA \\ C \end{bmatrix}$ has full rank for all $(s,t)\neq(0,0)$.
d) For all matrices V_i and V_f in standard form, the state x(i) where $i\in[n,N-n]$ can be uniquely determined from the outputs $y(j)$: $j\in[i-n,i+n-1]$.

It is also possible to define notions of weak reachability and observability which explicitly involve the boundary matrices V_i and V_f and to develop a theory of minimal realizations [1-2]. In addition, in [1] we develop methods for the recursive solution of (1) and develop several notions of stability for TPBVDSs.

III. The Optimal Smoother

Consider the system (1)-(2) together with the moise-corrupted observations

$$y(k) = Cx(k) + r(k) \qquad k=1,\ldots,N-1 \quad (9)$$

$$y_b = W_i x(0) + W_f x(N) + r_b \ . \quad (10)$$

Here r(k), r_b, u(k), and v are mutually independent, r_b is a zero mean, Gaussian random vector with covariance Π_b, and r(k) is a zero mean white Gaussian noise process with covariance R.

It can be shown [3] that the smoothed estimate $\hat{x}(k)$ satisfies the following TPBVDS

$$\mathcal{E}\begin{bmatrix}\hat{x}(k+1)\\ \lambda(k+1)\end{bmatrix} = \mathcal{A}\begin{bmatrix}\hat{x}(k)\\ \lambda(k)\end{bmatrix} + \begin{bmatrix}0\\ C'R^{-1}y(k)\end{bmatrix} \ , \ k=1,\ldots,N-1 \quad (11)$$

$$\mathcal{V}_i\begin{bmatrix}\hat{x}(1)\\ \lambda(1)\end{bmatrix} + \mathcal{V}_f\begin{bmatrix}\hat{x}(N)\\ \lambda(N)\end{bmatrix} = \mathcal{M}y_b \quad (12)$$

where

$$\mathcal{E} = \begin{bmatrix} E & -BQB' \\ 0 & -A' \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix} \quad (13)$$

and where $\mathcal{V}_i$, $\mathcal{V}_f$ and $\mathcal{M}$ are complicated matrices.

To compute the estimate we can use any of the recursive algorithms developed in [1-2]. One of these is the so-called two-filter solution in which the TPBVDS dynamics are decoupled into forward and backward recursions, followed by a correction to account for the boundary conditions. A necessary, but not sufficient, condition for stability of a TPBVDS is that it is *forward-backward stable*, i.e. a decoupling transformation can be found so that the forward and backward recursions are both stable.

In the case of the optimal smoother, it is shown in [3] that if the following generalized Riccati equations

$$\theta = A'(E\theta^{-1}E' + BQB')^{-1}A + C'R^{-1}C \tag{14}$$

$$\Psi = A(E'\Psi^{-1}E + C'R^{-1}C)^{-1}A' + BQB' \tag{15}$$

have positive definite solutions Ψ and θ then there exist invertible matrices M and N such that

$$M\mathcal{E}N^{-1} = \begin{bmatrix} I & 0 \\ 0 & A'S^{-1}E\theta^{-1} \end{bmatrix} \tag{16}$$

$$M\mathcal{A}N^{-1} = \begin{bmatrix} AT^{-1}E'\Psi^{-1} & 0 \\ 0 & I \end{bmatrix}. \tag{17}$$

Moreover, the eigenvalues of $AT^{-1}E'\Psi^{-1}$ and $A'S^{-1}E\theta^{-1}$ are inside or on the unit circle. Equation (3.5) is called the descriptor Hamiltonian equation and the above decomposition is the descriptor Hamiltonian diagonalization. Of course, we would like $AT^{-1}E'\Psi^{-1}$ and $A'S^{-1}E\theta^{-1}$ to be strictly stable. This occurs only when the descriptor Hamiltonian has no eigenmodes on the unit circle i.e. it is forward-backward stable.

Theorem 3:

If the system is forward-backward detectable and stabilizable (i.e. the modes on the unit circle are strongly reachable and strongly observable) then the corresponding smoother is forward-backward stable.

IV. Generalized Riccati Equations

In this section we study the generalized algebraic Riccati equation.

$$\varphi = A(E'\varphi^{-1}E + C'R^{-1}C)^{-1}A' + BQB'. \tag{18}$$

Theorem 4:

If (E,A,B) and (C,E,A) are strongly reachable and observable respectively then (18) has a unique positive definte solution.

The approach used to prove this theorem is similar to that in [6] for the standard Riccati equation. Details will be presented in a future paper. Existence proceeds as follows. From Theorem 3 and the fact that eigenmodes of the smoother occur in reciprocal pairs, we know that we can write

$$\begin{bmatrix} E & -BQB' \\ 0 & -A' \end{bmatrix}\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix}\begin{bmatrix} F \\ G \end{bmatrix}J \tag{19}$$

The proof then proceeds by first showing that F is invertible, then that $E'GF^{-1} + C'R^{-1}C > 0$ and finally that

$$\varphi = (A(E'GF^{-1} + C'R^{-1}C)^{-1}A' + BQB'); \tag{20}$$

satisfies (18).

To prove uniqueness, let φ_1 and φ_2 be two positive definite solutions of (18), let $\Delta\varphi = \varphi_1-\varphi_2$, and

$$T_i = E'\varphi_i^{-1}E + C'R^{-1}C \quad \text{for } i=1,2. \tag{21}$$

Some algebra then yields

$$\Delta\varphi = AT_1^{-1}E'\varphi_1^{-1}\Delta\varphi\varphi_2^{-1}ET_2^{-1}A'. \tag{22}$$

But $AT_1^{-1}E'\varphi_1^{-1}$ and $\varphi_2^{-1}ET_2^{-1}A'$ are strictly stable (see [3]); thus $\Delta\varphi = 0$.

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